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TWO RUNWAY INTERDICTION PROBLEMS (REVISITED)

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13. ABSTRACT (Maximum 200 words)

This report contains a description of two Monte Carlo computer programs and the associated statistical analysis for:

(a) Computing the single-run probability of interdicting an enemy rectangular field F, of given size, by NB bombs so that it is made useless as an airfield.

(b) Computing the average number of runs or sorties required for a prespecified probability of interdiction under the same conditions.

The bombs, each having a circular lethal area, with radius A, are directed at NB prespecified aim points with the actual hit points subject to a random normal aiming error and individual normal ballistic dispersions. F is interdicted if the bombing leaves no undamaged subrectangle within F, of given dimensions, with sides parallel to those of the field.

The statistical analysis and two Monte Carlo programs were originally described in a Naval Weapons Laboratory (now Naval Surface Warfare Center, Dahlgren Division [NSWCDD]) report by Dr. Milton P. Jarnagin, [1]. Our procedures obtain a near-optimum set of aim points. The choice of aim points is not discussed in [1]. Our results also differ in several other ways resulting in more conservative results but less efficiency than [1].

The computer programs and the graphics described in this report are written in FORTRAN 77 and MATLAB, respectively.

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FOREWORD

The interdiction problem treated in this report originated with Frederic C. Clodius (retired) of the Naval Weapons Laboratory (now Naval Surface Warfare Center, Dahlgren Division [NSWCDD]) in 1970. Lack of adequate computing speed and computer memory requirements then limited the generality and the extensive computing requirements of the simulation models for interdicting airfields. Readdressing the problems with the advanced computing capabilities of today allows for more generality in proposed models.

The author is indebted to Dr. Robert J. McDevitt (W25) and Dr. Demetrios Serakos (W25) for helpful discussions. David Bozicevich’s (CX7) scholarly editing of the report is appreciated.

This document was reviewed by John T. Beck, Head, Ballistic Missile Defense Systems Analysis Branch, and David S. Richardson, Head, Warfare Systems Development Division.

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CONTENTS

ILLUSTRATIONS

TABLES

GLOSSARY

The numeral at the end of each item is the page number where the item is first used.

(a)–Refers to the first of two problems considered, 1
(b)–Refers to the second of two problems considered, 2
(c)–Refers to five different subdivisions of side [(0,0)-(LF,0)], 4
(d)–Refers to five different subdivisions of side [(0,0)-(0,WF)], 4
$AC_{i,j}$–Identifies the subset of $C_{i,j}$ that contains aim points, 4
$C_{i,j}$–Identifies the rectangular cells that partition F, 4
$I_x(a, b)$–Incomplete beta function ratio with parameters a,b, 9
$\Gamma(x)$–Complete gamma function of x, 10
γ–=(e-d)/2, 8
λ–Level of confidence, 8
σ_{DA}–Standard deviation of deflection aiming error (y-direction), 1
σ_{DB}–Standard deviation of deflection ballistic dispersion of ijth bomb, 1
σ_{RA}–Standard deviation of range aiming error (x-direction), 1
σ_{RB}–Standard deviation of range ballistic dispersion of ijth bomb, 1
A–Lethal radius of a bomb, 1
AF–airfield or subrectangle, 1
EAC–Denotes an enlarged AC cell of polygonal shape, 6
EAF–Rectangle enclosing AF with sides ELR, EWR, 1
ELR–Length of rectangle EAF, ELR = LR + 2A, 1
EWR–Width of rectangle EAF, EWR = WR + 2A, 1
F–Original specified field or runway to be interdicted, 1
f–Total number of failures to interdict F in m sorties, 3
IC–Identifies the five subdivisions of (c), takes values 1-5, 4
IL–Integer part of LF/LR, 4
IW–Integer part of WF/WR, 4
JC–Identifies the five subdivisions of (d), takes values 1-5, 4
LA–Circular lethal area, with radius A, of a bomb, 1
LA2–Segment of circle, with circle bounding the LA, 6
LF–Input length of the field, 1
LR–Length of a subrectangle, AF, 1
LR1–Length of an AC cell, 4
m–Total number of sorties=s+f, 3
N(0,1)–Normal probability distribution with 0 mean, variance=1, 2

NB–Number of bombs dropped in a sortie, 1
NK–Number of sorties used in (a), 6
NP–The number of chords approximating a quarter circle of an EAC, 6
NP2–Denotes the number of vertices in an EAC, 4NP + 5, 6
p–=s/m, probability of interdicting F using m sorties, 3
RL–Fractional part of LF/LR, 4
RW–Fractional part of WF/WR, 4
s–Total number of successes to interdict F in m sorties, 3
WF–Input width of the field, 1
WR–Width of a subrectangle, AF, 1
WR1–Width of an AC cell, 4
XA(k)–x-coordinate of kth aim point, 2
XH(k)–x-coordinate of kth hit point, 2
YA(k)–y-coordinate of kth aim point, 2
YH(k)–y-coordinate of kth hit point, 2

I. INTRODUCTION

In this report we revisit the work by Dr. Milton P. Jarnagin, Jr., described in [1], entitled: *A Runway Interdiction Program.*

A rectangular field F of dimensions (LF x WF) with sides parallel to the X and Y axes is specified with its lower left-hand corner at $(0, 0)$ (see Figure 1). "Sortie" and "run" are the terms used to note the dropping of NB identical bombs directed at F. Interdiction of F means that it is impossible to find an enclosed undamaged subrectangle, AF, with sides parallel to the axes and of prespecified dimensions (LR x WR) after a sortie or sorties. Thus, according to [1], F is not interdicted if no bomb's lethal area penetrates the interior of a potential AF, where the lethal area, LA, of a bomb is the circle of radius A with center at the impact point of the bomb (see Figure 2). It is assumed everything is destroyed inside LA and nothing is destroyed by that bomb outside LA. On this basis, any bomb landing in a hypothetical rectangle, EF, enclosing F symmetrically with sides of lengths $\mathrm{ELF} = \mathrm{LF} + 2\,\mathrm{A}$, $\mathrm{EWF} = \mathrm{WF} + 2\,\mathrm{A}$ will result in some damage to F since its LA will intersect F except for a region at the corners of EF. Any bomb landing outside EF cannot contribute to the interdiction of F. In more detail, to determine whether F is interdicted requires that any AF must sustain damage from at least one bomb. Hence, any AF under examination is assumed to be symmetrically enclosed by a rectangle, EAF, with dimensions (ELR x EWR), where $\mathrm{ELR} = \mathrm{LR} + 2\,\mathrm{A}$, and $\mathrm{EWR} = \mathrm{WR} + 2\,\mathrm{A}$, so that AF is damaged if and only if the bomb's hit point is on or in the interior of EAF. This model, as given in [1], introduces a bias, since the distance from the corner of an EAF to the closest corner point of its AF is greater than A (see Figure 3, upper right hand corner). Thus the LA of a bomb near a corner of an EAF would not intersect AF, but the model assumes otherwise. This can result in optimistic probabilities of interdiction. The problem is eliminated in our model as described in the next section.

In Figure 2 below, an example of F not interdicted is shown, with NB = 4, since an undamaged AF exists. In Figure 3, with NB = 4, an example of F interdicted is shown since no undamaged AF can exist. In the figures, noting some of the variables, we have LF=6, WF=2, LR=2, WR=1, A=.25. Figures 1, 2, and 3 show an EAF.

Two problems are addressed, both treated by Monte Carlo techniques:

(a) Determine the probability of interdicting F with associated confidence limits by detonating a given number of identical bombs, NB, with given aim points and lethal circular area with radius A. The hit point of each bomb, with an associated aim point, is subject to the same normal random aiming error with assigned standard deviations in range and deflection, σ_{RA}, σ_{DA}, and normal ballistic dispersions with specified input standard deviations, $\sigma_{\mathrm{RB}}, \sigma_{\mathrm{DB}}$.

(b) Determine the average number of runs or sorties required to interdict F with a prespecified probability and associated confidence limits.

We proceed by briefly summarizing the Jarnagin model followed by a description of our Monte Carlo model (Section II), which differs significantly. Both approaches use the same classical statistical analysis.

The hit points for NB bombs, at the center of the circles as shown in Figures 2 and 3, have coordinates $(\mathrm{XH(k)}, \mathrm{YH(k)})$, $\mathrm{k} = 1, ..., \mathrm{NB}$. They are obtained in [1] as follows: The input aim points are specified by their coordinates $(\mathrm{XA(k)}, \mathrm{YA(k)})$. Then two normal random variables, $\psi_{\mathrm{X}}, \psi_{\mathrm{Y}}$, are generated from $\mathrm{N}(0,1)$ using a subroutine, RNOR, from [2], so that the aiming errors in range and deflection for every bomb are given by $\psi_{\mathrm{X}}\,\sigma_{\mathrm{RA}},\ \psi_{\mathrm{Y}}\,\sigma_{\mathrm{DA}}$, where σ_{RA} and σ_{DA} are standard deviation inputs, as noted above. The ballistic dispersion errors are modeled for each bomb by choosing $2\,\mathrm{NB}$ normal random variables $\theta_{\mathrm{X}}(\mathrm{k})$, $\theta_{\mathrm{Y}}(\mathrm{k})$ from $\mathrm{N}(0,1)$ using RNOR; so that range and deflection ballistic errors are given by $\theta_{\mathrm{X}}(\mathrm{k})\,\sigma_{\mathrm{RB}}$, $\theta_{\mathrm{Y}}(\mathrm{k})\,\sigma_{\mathrm{DB}}$, where σ_{RB} and σ_{DB} are standard deviation inputs. The hit points are given by

$$\mathrm{XH(k)} = \mathrm{XA(k)} + \psi_{\mathrm{X}}\,\sigma_{\mathrm{RA}} + \theta_{\mathrm{X}}(\mathrm{k})\,\sigma_{\mathrm{RB}} \tag{1}$$

$$\mathrm{YH(k)} = \mathrm{YA(k)} + \psi_{\mathrm{Y}}\,\sigma_{\mathrm{DA}} + \theta_{\mathrm{Y}}(\mathrm{k})\,\sigma_{\mathrm{DB}}, \quad \mathrm{k} = 1,\ \ldots,\ \mathrm{NB}. \tag{2}$$

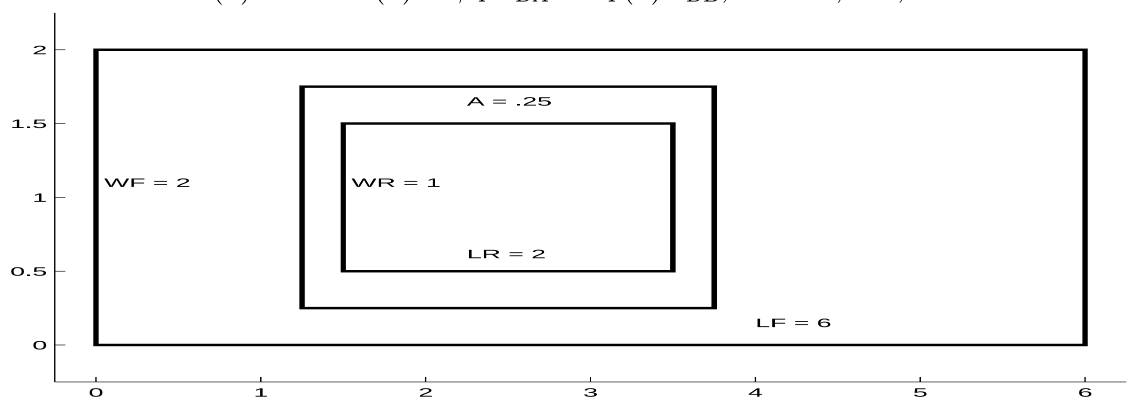

Figure 1. Shows F, A, AF, EAF.

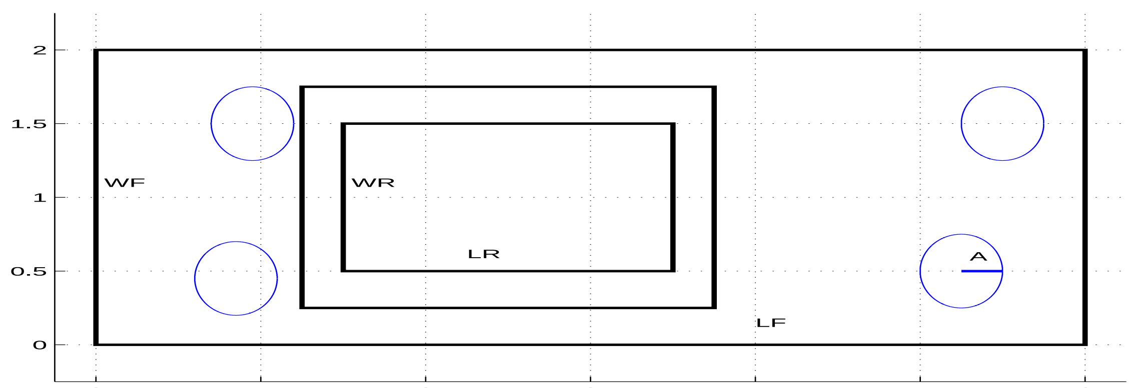

Figure 2. F not interdicted $(\mathrm{A} = .25, \mathrm{NB} = 4)$.

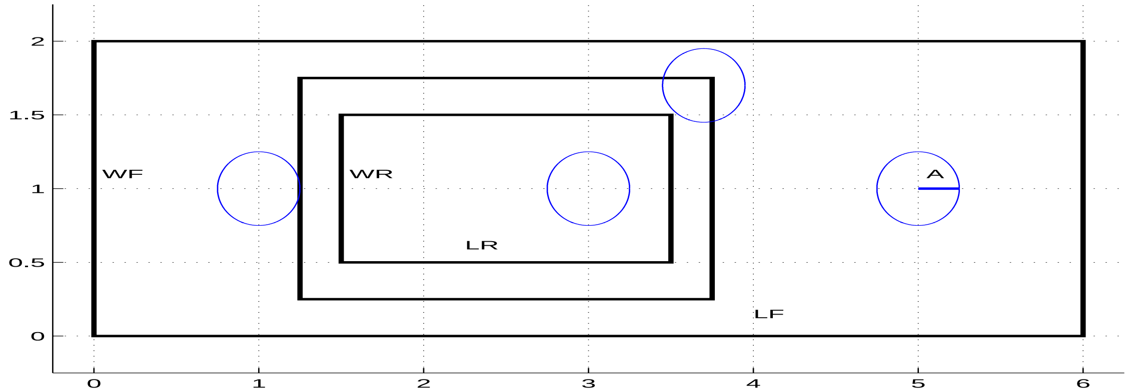

Figure 3. F interdicted.

It is assumed the bomb-depositing aircraft fly parallel to the X axis. The option of the aircraft flying at an angle β to the positive X axis is dealt with by a rotation of axes. Assume an orthogonal $\xi\eta$ set of axes with the same origin as the XY axes where the ξ axis is rotated counterclockwise through an angle β from the positive X axis. Then the kth hit point with coordinates $(\xi(\mathrm{k}), \eta(\mathrm{k}))$ is given, for the corresponding aim point $(\xi_{\mathrm{A}}(\mathrm{k}), \eta_{\mathrm{A}}(\mathrm{k}),$ by

$$\xi(\mathrm{k}) = \xi_{\mathrm{A}}(\mathrm{k}) + \psi_\xi\,\sigma_{\mathrm{RA}} + \theta_\xi(\mathrm{k})\,\sigma_{\mathrm{RB}} \tag{3}$$

$$\eta(\mathrm{k}) = \eta_{\mathrm{A}}(\mathrm{k}) + \psi_\eta\,\sigma_{\mathrm{DA}} + \theta_\eta(\mathrm{k})\,\sigma_{\mathrm{DB}}, \quad \mathrm{k} = 1,\,\ldots,\,\mathrm{NB}, \tag{4}$$

where the $2\,\mathrm{NB} + 2$ normal random numbers $\psi_\xi, \psi_\eta, \theta_\xi(\mathrm{k}), \theta_\eta(\mathrm{k})$ are chosen in the same way as those appearing in (1), (2). Thus, in the (X, Y) coordinates we have

$$\mathrm{XH}(\mathrm{k}) = \xi(\mathrm{k})\cos(\beta) - \eta(\mathrm{k})\sin(\beta) \tag{5}$$

$$\mathrm{YH}(\mathrm{k}) = \xi(\mathrm{k})\sin(\beta) + \eta(\mathrm{k})\cos(\beta), \quad \mathrm{k} = 1,\,\ldots,\,\mathrm{NB}. \tag{6}$$

This option is taken from [1] but is not included in our software. It would be a straightforward addition, if it were needed.

A dropping of NB bombs will be called a sortie or a run. Each sortie is assumed to have the same aim points, but, in general, with different bomb hit points. From [1], for a sortie, new sequences are constructed that reorder the XH(k) and YH(k) in increasing order of magnitude. Then, a set of elementary inequalities are evaluated based on these new sequences to decide whether F has been interdicted. If the answer is "yes," the sortie is called a success, s; otherwise, it is called a failure, f. If m sorties are run, then the ratio $\mathrm{p} = \mathrm{s}/\mathrm{m}$ is taken as an estimate of the probability of interdicting F by a single sortie (based on m sorties). Then, as described in [1], confidence limits associated with p and m are established. We defer this part of the discussion until Section III since we will use the same statistical classical analysis.

The Monte Carlo procedure as just described is very efficient and fast, but it has several drawbacks. Namely,

1. As mentioned in the Introduction, a bomb hit point near the corner of an EAF would still allow credit to the enclosed AF in spite of the fact that the LA for that particular bomb would not intersect the AF. This can lead to optimistic values for p.
2. No a priori plans on optimum aiming of the bombs in a sortie are discussed.
3. The requirement that if any part of a bomb's LA intersects an AF, no matter how little, that AF is considered damaged also contributes to optimistic estimates for p.

II. OUR MODEL

Consider F made up of a rectangle, sides parallel to the X and Y axes with its lower left-hand corner positioned at (0,0). In our model, F is partitioned into rectangular cells with sides parallel to the X and Y axes. Five different subdivisions of side (0,0)-(LF,0), not necessarily equal, are used and referred to as (c), and five different subdivisions of side (0,0)-(0,WF) are used, referred to as (d), to generate the partitioning. Thus, a total of 25 different partitionings are considered. The rectangular cells are identified for a particular partitioning as shown in Figure 4, but in general, are not of equal size. We will describe (c) understanding that (d) uses the same rules with appropriate changes of variables such as WF for LF. Before discussing the five subdivision cases under (c), some additional notation is introduced. Let

IL = integer part of LF/LR (LF and LR defined in the previous section).

IW = integer part of WF/WR (WF and WR defined in the previous section).

RL = LF − IL ∗ LR, fractional part of LF/LR.

RW = WF − IW ∗ WR, fractional part of WF/WR.

$C_{i,j}$ identifies the rectangular cells that are made up from the partitioning of F with $i = 1, ..., IWC$, $j = 1, ..., ILC$ (See an example in Figure 4.).

$AC_{i,j}$ denotes the subset of the $C_{i,j}$ cells each containing an aim point with $i = 2, ..., IWC - 1$, $j = 2, ..., ILC - 1$ (See an example in Figure 6 where the aim points are indicated with small filled circles at the center of each AC cell.).

The size of the AC cells is (LR1xWR1). IC=1,2,3,4,5, identifies the five subdivisions of (c). JC=1,2,3,4,5, identifies the five subdivisions of (d).

We now describe some of the IC, JC cases, with emphasis on the AC cells. More detail is given by assigning numerical values to the defining parameters. The idea behind the generation of the AC cells is to make them as large as possible, thus increasing the possibility of a success, but small enough so that no AF can exist if all the AC cells of a partitioning are interdicted. We assume that A < WR1 < LR1.

Let LF=6, WF=2, A=.25. Initial value for i and j of $AC_{i,j}$ is 2. Then

$\underline{\text{IC} = 1,\ \text{JC} = 1}$, IL > 1, IW > 1,   Figure 6
LR $= 2$, WR $= 1$, IL $= 3$, IW $= 2$, RL $= 0$, RW $= 0$, LR1 $=$ LR/2 $= 1$,
WR1 $=$ WR/2 $= .5$, ILC $= 2$IL $= 6$, IWC $= 2$IW $= 4$, NB $= 4$(IL $- 1$)(IW $- 1$) $= 8$.

$\underline{\text{IC} = 2,\ \text{JC} = 2}$, IL > 1, IW > 1, RL $<$ LR/2, RW $<$ WR/2, Figure 7
LR $= 2.5$, WR $= .85$, IL $= 2$, IW $= 2$, RL $= 1$, RW $= .3$, LR1 $=$ LR/2 $= 1.25$,
WR1 $=$ WR/2 $= .425$, ILC $= 2$IL$+1 = 5$, IWC $= 2$IW$+1 = 5$, NB $=$ (2IL$-$1)(2IW$-$1) $= 9$.

$\underline{\text{IC} = 3,\ \text{JC} = 3}$, IL > 1, IW > 1, RL $\geq$ LR/2, RW $\geq$ WR/2, Figure 8
LR $= 2.25$, WR $= .75$, IL $= 2$, IW $= 2$, RL $= 1.5$, RW $= .50$, LR1 $=$ LR/2 $= 1.125$,
WR1 $=$ WR/2 $= .375$, ILC $= 2$(IL $+ 1$) $= 6$, IWC $= 2$(IW $+ 1$) $= 6$, NB $= 4$IL $*$ IW $= 16$.

$\underline{\text{IC} = 4,\ \text{JC} = 4}$, IL $= 1$, IW $= 1$, RL $>$ LR/2, RW $>$ WR/2, Figure 9
LR $= 3.5$, WR $= 1.2$, IL $= 1$, 1W $= 1$, RL $= 2.5$, RW $= .8$, LR1 $=$ LR/2 $= 1.75$,
WR1 $=$ WR/2 $= .60$, ILC $= 4$IL $= 4$, IWC $= 4$IW $= 4$, NB $= 2$IL $* 2$IL $= 4$.

$\underline{\text{IC} = 5,\ \text{JC} = 5}$, IL $= 1$, IW $= 1$, RL $\leq$ LR/2, RW $\leq$ WR/2, Figure 10
LR $= 4$, WR $= 1.65$, IL $= 1$, IW $= 1$, RL $= 2$, RW $= .35$, LR1 $=$ LR $-$ RL $= 2$,
WR1 $=$ WR $-$ RW $= 1.30$, ILC $= 2$IL $+ 1 = 3$, IWC $= 2$IW $+ 1 = 3$, NB $=$ IL $*$ IW $= 1$.

$\underline{\text{IC} = 2,\ \text{JC} = 3}$, IL > 1, IW > 1, RL $<$ LR/2, RW $\geq>$ WR/2, Figure 11
LR $= 2.5$, WR $= .75$, IL $= 2$, IW $= 2$, RL $= 1$, RW $= .50$, LR1 $=$ LR/2 $= 1.25$,
WR1 $=$ WR/2 $= .375$, ILC $= 2$IL$+1 = 5$, IWC $= 2$(IW$+1$) $= 6$, NB $=$ (2IL$-$1)(2IW) $= 12$.

Also, it should be noted that there exist chances that a bomb falls outside an EAC cell and contributes to a success. Such events are not recognized in our model, thus tending to a conservative estimate for the probability of the interdiction of F. An example is shown in Figure 5 below.

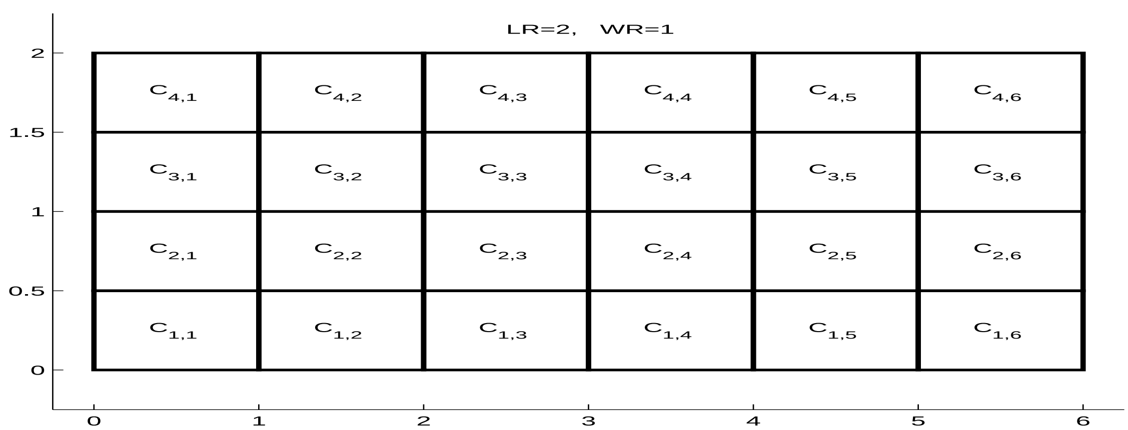

Figure 4. Notation for a celled field.

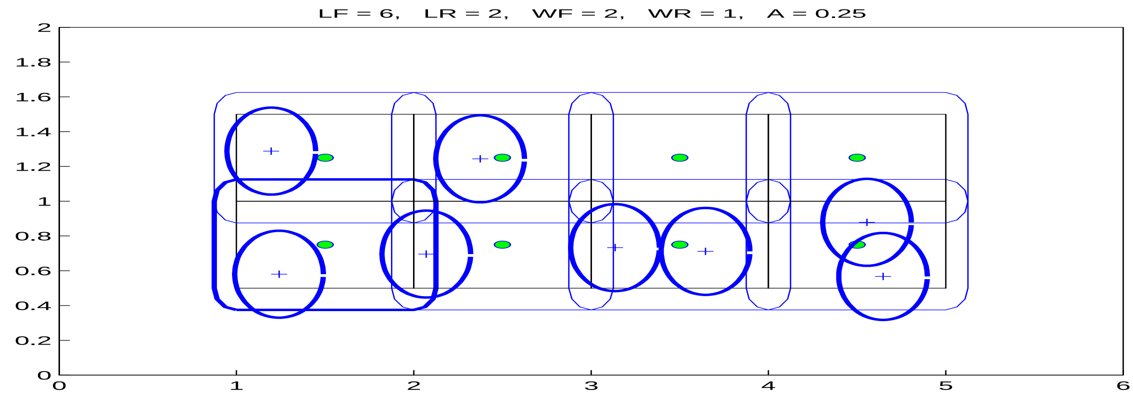

Figure 5. An interdicted field with an empty AC cell, $C_{3,4}$.

Let LA2 denote the smaller segment of a circle centered at the origin with radius A that is bounded by the chord $(-\sqrt{3}/2\,\mathrm{A}, \mathrm{A}/2) - (\sqrt{3}/2\,\mathrm{A}, \mathrm{A}/2)$ and the circular arc subtended by the chord. We say an AC cell is *interdicted* if at least LA2 of a bomb intersects the cell. This is ensured if every AC cell is surrounded by a rectangle with rounded corners, namely quarter circles, separated from its AC cell by a constant distance of A/2. In our model, a quarter circle is approximated by NP equal, in length, chords subtending the arc. Denote the polygonal figure by EAC (or $\mathrm{EAC}_{\mathrm{i,j}}$), so that, if a bomb falls on or inside an EAC, the associated AC is considered interdicted. This polygonal approximation, with $\mathrm{NP2} = 4\mathrm{NP} + 5$ vertices, leads to a slightly pessimistic probability of success. Note also the overlapping of the EAC's. Thus, one bomb can interdict more than one EAC. Figures 6-11 show the EAC's for particular IC, JC with $\mathrm{EAC}_{2,\,2}$ depicted in bold. Hence, if all EAC's are interdicted, then F is interdicted, and our sortie is a success.

Since the EAC's are polygons, we use a routine from [2], LOCPT, which determines for an input polygon PO, in the plane, whether a point $(\mathrm{xo}, \mathrm{yo})$ is outside PO. Thus, each EAC, for a particular set IC, JC, is examined by LOCPT to determine which of the hit points, as given by $(\mathrm{XH(k)}, \mathrm{YH(k)})$, $\mathrm{k} = 1, 2, ..., \mathrm{NB}$, (See (1), (2)), interdict one or more of the EAC's.

An added feature of our software is that we can specify a sequence of sorties, NK, in a single run keeping track of the damaged parts of F. Thus, if NK=3 and if, after the first sortie, F is interdicted, then the run is declared a success; however, if F is not interdicted, a second sortie is made, and if F is interdicted then, a success is declared. However, if again F is not interdicted, a final sortie is made before a run is declared a success or failure. In [1], this option is not available or discussed.

The statistical analyses that follow in the next two sections are taken directly from [1]. They are also given in [3] and [4].

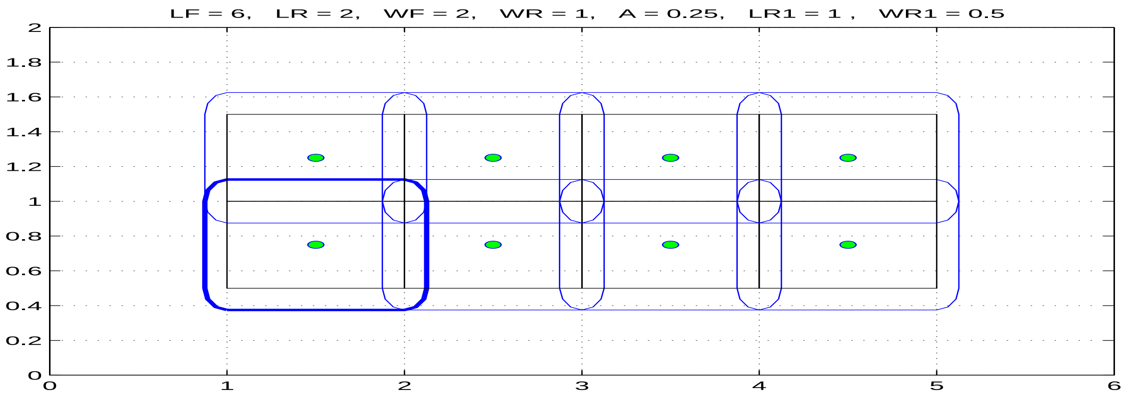

Figure 6. IC=1, JC=1. An example of a partitioning of F.

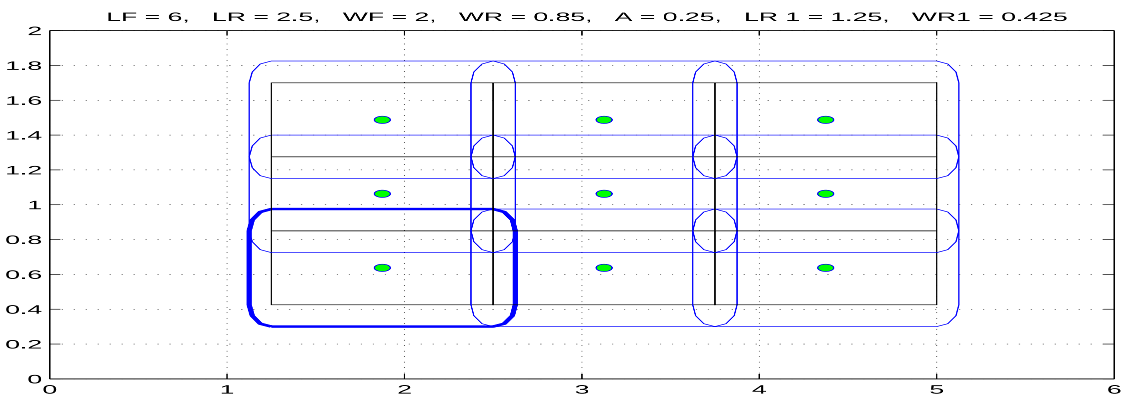

Figure 7. IC=2, JC=2. An example of a partitioning of F.

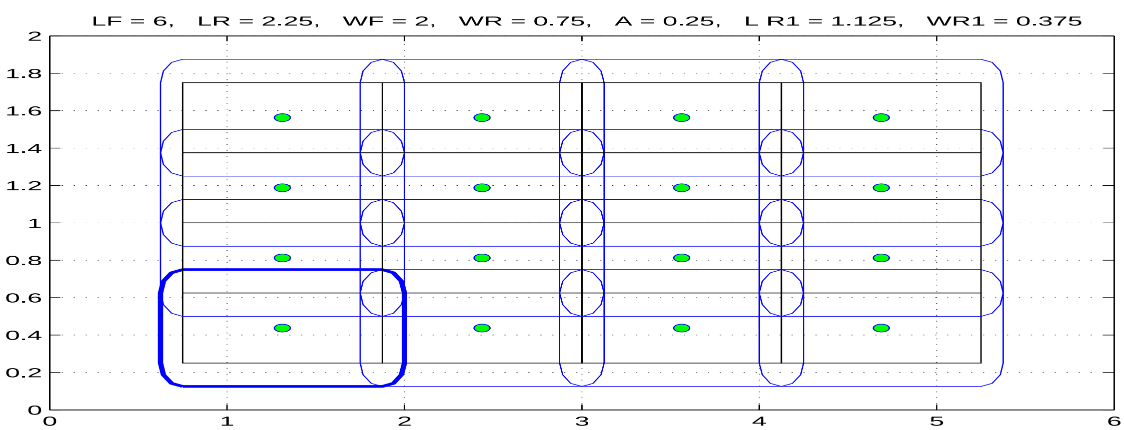

Figure 8. IC=3, JC=3. An example of a partitioning of F.

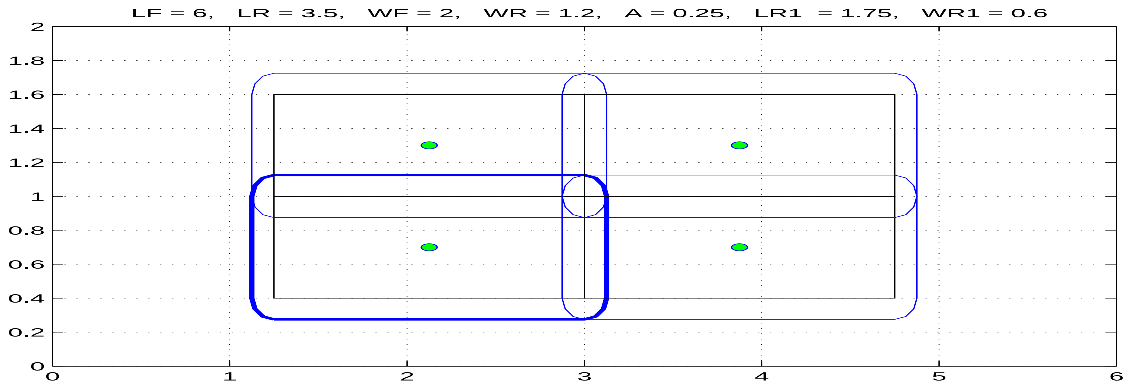

Figure 9. IC=4, JC=4. An example of a partitioning of F.

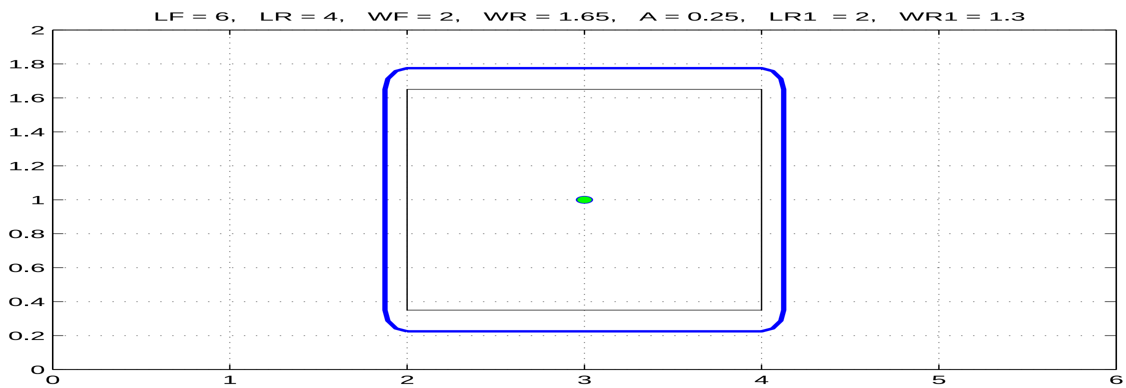

Figure 10. IC=5, JC=5. An example of a partitioning of F.

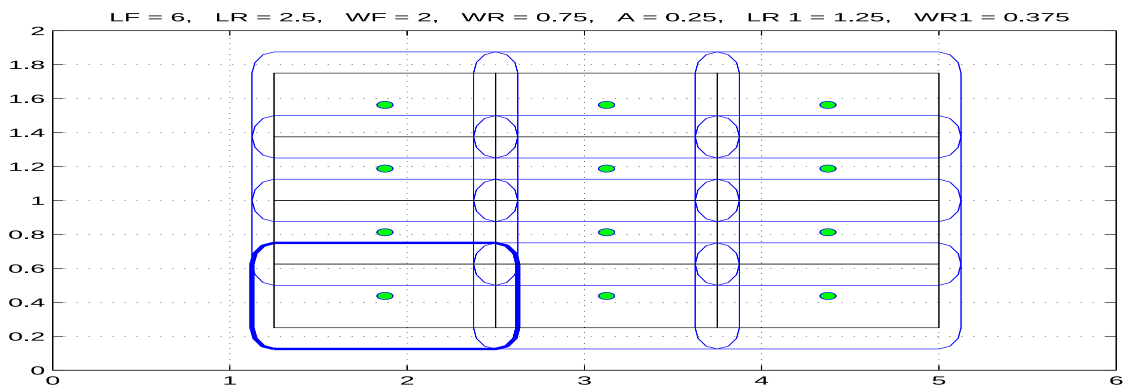

Figure 11. IC=2, JC=3. An example of a partitioning of F.

III. STATISTICAL ANALYSIS FOR PROBLEM (a)

The input for Problem (a) is: LF, WF, LR, WR, A, σ_{RA}, σ_{DA}, σ_{RB}, σ_{DB}, λ, γ. The last two will be defined below. Once the IC, JC partitioning of F is established, NB is known. Then NB aim points are generated, one at the center of each EAC cell.

A sortie or run means that NB bombs are dropped with one aimed at each aim point as shown in Figures 6-11. Then, using a normal random number generator RNOR, from [2], the four sigma inputs (standard deviations) are used to determine the actual hit points of each bomb as given by (1) and (2). For a success, each EAC must be interdicted by at least one bomb; otherwise, a failure to interdict has occurred. The decision as to whether a particular polygonal EAC has been hit is determined by the subroutine LOCPT from [2], which reports if the kth hit point is inside, on the boundary, or outside any of the EAC's. In total, m runs or sorties are made where s is equal to the number of successes and f is equal to the number of failures so that $m = s + f$.

The quantity $p = s/m$ is taken as an estimate of the probability of interdicting F. By computing confidence limits on p, we can establish a level of confidence in p by running enough sorties. A value of γ is chosen to achieve a given level of confidence. For example, if $\gamma = 0.05$, then we are aiming for a confidence level of $1 - \gamma = 0.95\%$. The lower and upper confidence limits d, e, are specified by the input parameter λ so that $0 < p - d \leq \lambda$, and $0 < e - p \leq \lambda$, or with $1 - \gamma$ confidence, the true but unknown probability p_0 is in (d, e).

In order to determine d, e, with $q \equiv 1 - p$, we will need the two relations that follow;

$$\sum_{i=s}^{m} \binom{m}{i} p^i q^{m-i} = I_p(s,\, m - s + 1), \quad [5, \text{page } 975], \tag{7}$$

$$\sum_{i=0}^{s} \binom{m}{i} p^i q^{m-i} = 1 - I_p(s + 1,\, m - s) = I_q(m - s,\, s + 1). \tag{8}$$

Note (8) follows from (7) since

$$\sum_{i=0}^{s} \binom{m}{i} p^i q^{m-i} + \sum_{i=s+1}^{m} \binom{m}{i} p^i q^{m-i} = 1. \tag{9}$$

The quantity $I_x(a,\, b)$ is known as the incomplete beta function ratio. It is evaluated by the subroutine ISUBX from [2], [6]. The expression on the left of (7) represents the probability of s or more successes in m trials. The expression on the left of (8) represents the probability of s or fewer successes in m trials. The algorithm for d and e proceeds as follows:

If $p \leq .5$, then sufficient sorties are run (summing successes, s, and failures, f) with a clean field on each one until $I_{q-\lambda}(m - s, s + 1) < \gamma/2$, then if $p \leq \lambda$ set $d = 0$, and compute e and exit, else continue the sorties until $I_{p-\lambda}(s, m - s + 1) < \gamma/2$, then compute d and e and exit.
If $p > .5$, then sufficient sorties are run, (summing successes, s, and failures, f) with a clean field on each one until $I_{p-\lambda}(s, m - s + 1) < \gamma/2$, then if $p \geq 1 - \lambda$ set $e = 1$, and compute d and exit, else continue the sorties until $I_{q-\lambda}(m - s, s + 1) < \gamma/2$, then compute d and e and exit, [3], [4].

In order to compute d, let $f(p) = I_p(s, m-s+1) - \gamma/2$ [1]. Then d=p is obtained when f(p)=0. The use of the Newton-Raphson procedure to find d requires the derivative of f with respect to p, $f'(p)$, i.e.,

$$f'(p) = \frac{\Gamma(m+1)}{\Gamma(s)\,\Gamma(m-s+1)} p^{s-1}\, q^{m-s} > 0, \tag{10}$$

where $\Gamma(x)$ represents the complete gamma function of x, [5, page 255]. Start with $p = d_0 = \left(\frac{s-1}{m-1}\right)$ [2], then in general $d_{n+1} = d_n - f(d_n)/f'(d_n), n = 0, 1, \ldots$. Convergence occurs when $|(d_{n+1} - d_n)/d_n| < \text{eps}$, where eps is a small positive number, say 5e-5. In order to compute e, let $g(p) = I_q(m-s, s+1) - \gamma/2$. Then e=p is obtained when g(p)=0. The use of the Newton-Raphson procedure to find e requires the derivative of g with respect to p, $g'(p)$, i.e.,

$$g'(p) = -\frac{\Gamma(m+1)}{\Gamma(s+1)\,\Gamma(m-s)} p^{s}\, q^{m-s-1} < 0. \tag{11}$$

Start with $p = e_0 = \left(\frac{s}{m-1}\right)$ [3], then, in general, $e_{n+1} = e_n - g(e_n)/g'(e_n), n = 0, 1, \ldots$. Convergence occurs when $|(e_{n+1} - e_n)/e_n| < \text{eps}$, where eps is a small positive number, say 5e-5. See Figure 12.

The subroutine INVBETA is used to find p given I_p. It is stored in INVB.FOR. The functions in (10) and (11) are computed using the subroutine BRCOMP. The latter routine is taken from [2]. The total Fortran software is contained in four files: JARN1.FOR INVB.FOR POLY2X.FOR, OINB.FOR. POLY2X contains the subroutine ILIW that is used to partition F and set the parameters IC, JC. OINB.FOR contains supporting software taken from [2]. In addition, JARN1.FOR contains code that generates input data for a MATLAB routine ILIWPIC.M to plot a figure such as shown in Figures 6-11 based on the particular value of IC and that of JC.

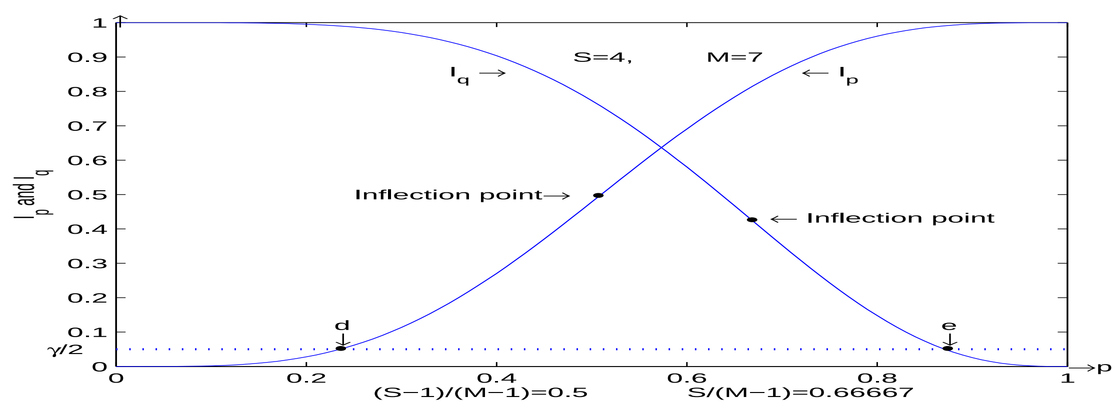

Figure 12. An example of an I_p curve and an I_q curve.

[1]Note p here has a different meaning than s/m.
[2]Note $f''(d_0) = 0$. The curvature is positive for $p < d_0$, so the Newton iterates converge.
[3]Note $g''(e_0) = 0$. The curvature is positive for $p > e_0$, so the Newton iterates converge.

IV. NUMERICAL RESULTS FOR PROBLEM (a)

In this section, some numerical results are given for problem (a) in Table 1. Results are tabulated for six different IC, JC cases. The input common to all cases is: LF=6, WF=2, A=.25, NP=3, $\sigma_{\mathrm{RA}} = .75$, $\sigma_{\mathrm{DA}} = .50$, $\sigma_{\mathrm{RB}} = .25$, $\sigma_{\mathrm{DB}} = .20$, with a confidence interval and level set at $\lambda = .05$, and $\gamma = .05$, respectively.

In the first line of output, under the m columns, we show 300(3) and 220(44). In the first case, the interpretation is that runs of 100 sorties 3 times, accumulating s's and f's, were required before the appropriate inequalities in the boxed remarks of the previous section were satisfied. A lower confidence limit of d = 0.083, and an upper confidence limit of e = 0.159, were obtained. In the second case, 44 sequences of 5 sorties each were run before the appropriate inequalities of the boxed section were satisfied. A lower confidence limit of d = 0.075, and an upper confidence limit of e = 0.163 were found. In the second line of output, 5000(1) indicates a single set of 5000 sorties was sufficient to satisfy the appropriate inequalities.

Table 1. Various Cases for Problem (a)
Confidence Limits d, e, and Probability of Interdiction, p

NK	m	d	p	e	NK	m	d	p	e
LR=2, WR=1, IC=1, JC=1, NB=8, Figure 6									
1	300(3)	.083	.117	.159	1	220(44)	.075	.114	.163
1	5000(1)	.099	.108	.117	1	10000(1)	.102	.108	.115
3	450(3)	.639	.684	.727	3	370(74)	.639	.689	.736
3	5000(1)	.656	.669	.682	3	10000(1)	.664	.673	.682
LR=2.5, WR=.85, IC=2, JC=2, NB=9, Figure 7									
1	300(3)	.100	.137	.181	1	245(49)	.095	.135	.184
1	5000(1)	.106	.115	.124	1	10000(1)	.105	111	.117
3	400(4)	.658	.705	.749	3	360(72)	.658	.708	.755
3	5000(1)	.709	.721	.734	3	10000(1)	.714	.723	.731
LR=2.25, WR=.75, IC=3, JC=3, NB=16, Figure 8									
1	200(2)	.000	.035	.071	1	115(23)	.000	.026	.074
1	5000(1)	.000	.047	.054	1	10000(1)	.000	.049	.053
3	400(4)	.632	.680	.725	3	375(75)	.630	.680	.727
3	5000(1)	.657	.670	.683	3	10000(1)	.661	.670	.680
LR=3.50, WR=1.20, IC=4, JC=4, NB=4, Figure 9									
1	400(4)	.272	.317	.366	1	370(74)	.267	.314	.363
1	5000(1)	.283	.295	.308	1	10000(1)	.288	.297	.306
3	250(5	.824	.872	.911	3	240(48)	.822	.871	.911
3	5000(1)	.871	.881	.889	3	10000(1)	.872	.878	.885
LR=4.00, WR=1.65, IC=5, JC=5, NB=1, Figure 10									
1	400(4)	.637	.685	.730	1	375(75)	.633	.683	.730
1	5000(1)	.697	.710	.722	1	10000(1)	.705	.714	.722
2	200(2)	.873	.920	.954	2	190(38)	.867	.916	.951
2	5000(1)	.910	.918	.925	2	10000(1)	.915	.920	.925
LR=2.50, WR=.75, IC=2, JC=3, NB=12, Figure 11									
1	200(2)	.054	.090	.139	1	190(38)	.053	.089	.139
1	5000(1)	.071	.078	.086	1	10000(1)	.074	.079	.085
3	400(4)	.647	.695	.740	3	370(74)	.645	.695	.741
3	5000(1)	.695	.708	.721	3	10000(1)	.705	.714	.723

V. STATISTICAL ANALYSIS FOR PROBLEM (b)

In this section we consider problem (b), first stated in the Introduction, namely, determine the expected number of sorties to interdict F with associated confidence limits. The idea is to generate a set of sequences, each sequence made up of a number of sorties to interdict F. All sorties have the same aim points. For example, starting with a clean target F, NB bomb hit points, determine if F is interdicted; if not, then keeping track of the hit points, another sortie is made with the same aim points, but with different hit points. This is continued until F is interdicted. Say eight sorties are required, then $N_1 = 8$. A clean target is used to repeat the above procedure with the ith sequence yielding a value of N_i. As a result, a mean and standard deviation are generated from k sequences, that is

$$M_k = \sum_{i=1}^{k} N_i/k, \tag{12}$$

$$\sigma_k = \left[\frac{1}{k-1}\sum_{i=1}^{k}(N_i - M_k)^2\right]^{\frac{1}{2}}. \tag{13}$$

The value of k is determined by using the Student t test to conclude when a prespecified confidence level is achieved, [3], [4]. In [1], two parameters, γ and λ, are introduced for this purpose. The number $1-\gamma$ specifies the level of confidence that the true expected number of sorties to interdict, M_0, is in the interval $(M_k - \lambda, M_k + \lambda)$. For example, if $\gamma = .05$, $\lambda = 2$, and $M_k = 4.2$, then we have that $1-\gamma$, implying a 95% confidence that the true mean, M_0, is in the interval $(4.2-2, 4.2+2)$. In applying the t test, the following quantities are used:

$$t_0^2 = (k\lambda^2)/\sigma_k^2, \tag{14}$$

$$x_0 = (k-1)/(k-1+t_0^2), \tag{15}$$

$$p_0 = 1 - I_x[(k-1)/2, 1/2]. \tag{16}$$

The quantity p_0 represents the probability that t of the Student t distribution satisfies the inequality $t \le t_0 = \lambda\sqrt{k}/\sigma_k$. The derivation of these equations is discussed in [1], [7, see page 4]. The numerical procedure, using (14)-(16), with the zero subscript dropped, is as follows: the number of sequences, k, is run a minimum number of times, say ten; thereafter, k is increased by some constant increment, say one, until $p \ge 1-\gamma$. At that point, x is found by using Newton-Raphson to solve the equation

$$1 - I_x[(k-1)/2, 1/2] = 1 - \gamma.^4 \tag{17}$$

[4] Figure 13 shows the graphical relationship of (17). So that, starting with $(1+x)/2$, with x from (15), convergence of Newton-Raphson is assured, since $\phi'(x)$ and $\phi''(x)$ from (18) are shown in [1] to always be negative, $k \ge 6$. The subroutine INVBE is used to find x from (18). It is stored in INVB.FOR.

or

$$\phi(\mathrm{x}) = (1 - \mathrm{I_x}[(\mathrm{k}-1)/2, 1/2]) - (1-\gamma) = \gamma - \mathrm{I_x}[(\mathrm{k}-1)/2, 1/2]. \tag{18}$$

Knowing x, (15) is solved for t^2, namely

$$\mathrm{t}^2 = (\mathrm{k}-1)(1-\mathrm{x})/\mathrm{x} \tag{19}$$

Then, solving (14) for λ gives λ'

$$\lambda' = \mathrm{t}\sigma_\mathrm{k}/\sqrt{\mathrm{k}} \tag{20}$$

Thus, we have arrived at the result that, with an assigned γ say $\gamma = .05$, the true expected value $\mathrm{M_0}$ lies in the interval $(\mathrm{M_k} - \lambda', \mathrm{M_k} + \lambda')$ with a confidence level of $1 - \gamma = 95(\%)$.

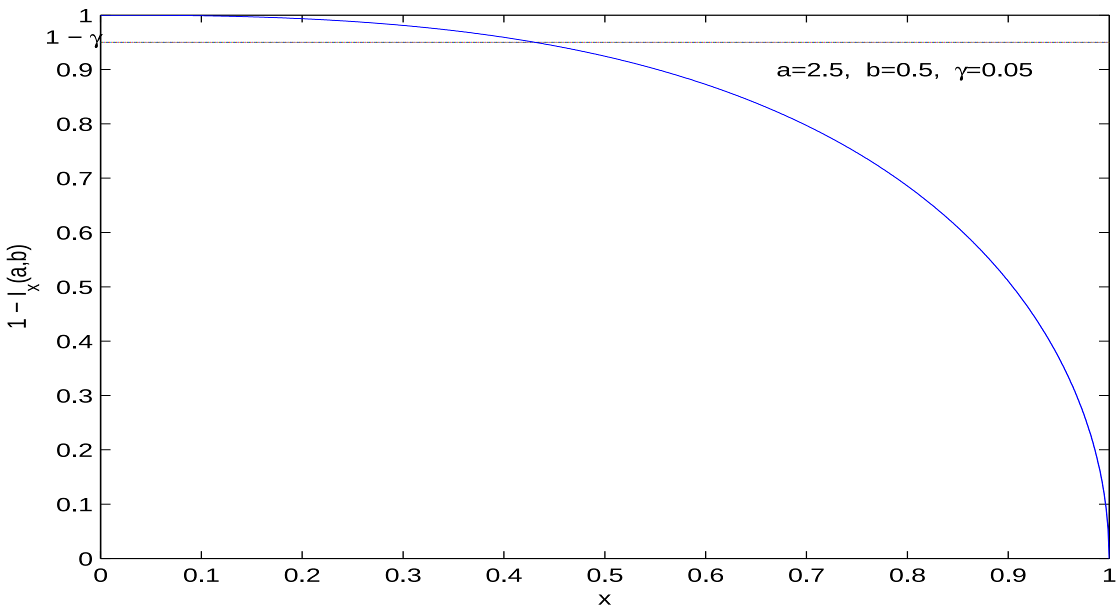

Figure 13. A graph of $1 - \mathrm{Ix}(\mathrm{a}, \mathrm{b})$ with also line $\mathrm{y} = 1 - \gamma$.

VI. NUMERICAL RESULTS FOR PROBLEM (b)

In this section some numerical results are given for problem (b). Results are tabulated for six different IC, JC cases. The input common to all cases is: LF=6, WF=2, A=0.25, $\sigma_{\mathrm{RA}} = 0.75$, $\sigma_{\mathrm{DA}} = 0.50$, $\sigma_{\mathrm{RB}} = 0.25$, $\sigma_{\mathrm{DB}} = 0.20, \mathrm{NP} = 3$ with a confidence level set at 95% and $\gamma = 0.05$, with a confidence interval parameter $\lambda = 1.0$. In the first line of output in Table 2, one sequence was run made up of 100 sorties, which was more than enough sorties to satisfy $\mathrm{p} \geq 1 - \gamma$ and achieve the 95% confidence level. The first line of output in the table refers to 100 sequence (k=100), with $\mathrm{M_k} = 3.04$ with $\lambda' = 0.29$. For the same input, three more results are listed for the same input with $\mathrm{k} = 1000, 5000, 10000$. The results in the table show that it is advantageous to run a very large number of sequences rather than start at a small number, say 10, and increase by an increment of one until the above inequality is satisfied.

Table 2. Various Cases for Problem (b) Confidence Level $\gamma = 0.05$, and Confidence Interval, $\lambda = 1.0$									
LR=2, WR=1, A=0.25, IC=1, JC=1, NB=8, Figure 6									
k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k	k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k
100	2.75	3.04	3.33	1.442	1000	3.10	3.19	3.29	1.526
5000	3.11	3.15	3.19	1.528	10000	3.11	3.14	3.17	1.557
LR=2.5, WR=.85, IC=2, JC=2, NB=9, Figure 7									
100	2.67	2.97	3.27	1.521	1000	2.90	2.99	3.08	1.464
5000	2.92	2.96	2.99	1.389	10000	2.91	2.94	2.97	1.375
LR=2.25, WR=.75, IC=3, JC=3, NB=16, Figure 8									
100	2.47	2.75	3.03	1.388	1000	2.84	2.92	3.01	1.385
5000	2.86	2.90	2.94	1.365	10000	2,86	2.89	2.91	1.379
LR=3.50, WR=1.20, IC=4, JC=4, NB=4, Figure 9									
100	1.88	2.11	2.34	1.154	1000	2.16	2.23	2.31	1.186
5000	2.17	2.20	2.24	1.157	10000	2.19	2.21	2.23	1.164
LR=4.00, WR=1.65, IC=5, JC=5, NB=1, Figure 10									
100	1,28	1.45	1.62	0.833	1000	1.39	1.44	1.49	0.797
5000	1.38	1.41	1.43	0.738	10000	1.38	1.39	1.41	0.737
LR=2.50, WR=.75, IC=2, JC=3, NB=12, Figure 11									
100	2.63	2.87	3.11	1.195	1000	3.00	3.08	3.17	1.376
5000	3.00	3.03	3.07	1.361	10000	2.98	3.01	3.04	1.344

In [1] the data for two cases are discussed. The aim points are not specified. The input data of [1], besides that shown in Table 3, are: LF = 6000, WF = 200, NB = 4, A = 20, k = 100, $\sigma_{\mathrm{RA}} = 250$, $\sigma_{\mathrm{DA}} = 125$, $\sigma_{\mathrm{RB}} = 40$, $\sigma_{\mathrm{DB}} = 20$. Our results for comparison are given in Table 3 (see Figure 14). The results of [1] are $M_k = 4.73$, $\lambda' = .670$, $\sigma_k = 3.125$ in one case. In the second case with the same input, the reference lists $M_k = 4.01$, $\lambda' = .405$, $\sigma_k = 2.042$. The results in [1] indicate that, on the average, 16-19 bombs are required for interdiction, whereas our results would indicate 21-22 bombs would be required. However, at least LA2 of a bomb must intersect an AF for its interdiction (see page 5), whereas in the case of [1], if any of the LA intersects AF, no matter how little, interdiction of that AF occurs. Thus, a more appropriate input A for comparison would be, in our case, to set A=30 instead of 20. From Table 3 one would conclude, on the average, around 19 bombs would be required by our results.

Table 3. Various Cases for Problem (b) (Comparison of Results with Results in [1]) Confidence Level $\gamma = 0.05$, and Confidence Interval, $\lambda = 1.0$									
LR=4200, WR=50, A=30, IC=5, JC=5, NB=6, Figure 14									
k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k	k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k
100	2.85	3.22	3.59	1.840	1000	3.15	3.28	3.40	1.983
5000	3.22	3.28	3.33	1.942	10000	3.23	3.27	3.30	1.934
LR=4200, WR=50, A=20, IC=5, JC=5, NB=6, Figure 10									
100	3.21	3.60	3.99	1.969	1000	3.64	3.77	3.90	2.150
5000	3.71	3.77	3.83	2.173	10000	3.75	3.79	3.83	2.179

Finally a conjecture is offered that, in the [1] input data, rather than RA $= 250$, $\sigma_{\mathrm{DA}} = 125$, $\sigma_{\mathrm{RB}} = 40$, $\sigma_{\mathrm{DB}} = 20$, perhaps the more realistic values, namely RA $= 250$, $\sigma_{\mathrm{DA}} = 40$, $\sigma_{\mathrm{RB}} = 125$, $\sigma_{\mathrm{DB}} = 20$ were intended. Our results in this case are shown in Table 4, indicating a need on the average of only 11-12 bombs for interdiction of F.

Table 4. Various Cases for Problem (b) (Comparison of Results with Modified Results in [1]) Confidence Level $\gamma = 0.05$, and Confidence Interval, $\lambda = 1.0$									
LR=4200, WR=50, A=30, IC=5, JC=5, NB=6, Figure 14									
k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k	k	$M_k - \lambda'$	M_k	$M_k + \lambda'$	σ_k
100	1.724	1.88	2.04	0.782	1000	1.77	1.83	1.88	0.869
5000	1.79	1.81	1.83	0.861	10000	1.77	1.78	1.80	0.854
LR=4200, WR=50, A=20, IC=5, JC=5, NB=6, Figure 14									
100	1.86	2.01	2.16	0.772	1000	2.08	2.14	2.20	1.010
5000	2.09	2.11	2.14	0.966	10000	2.09	2.11	2.13	0.959

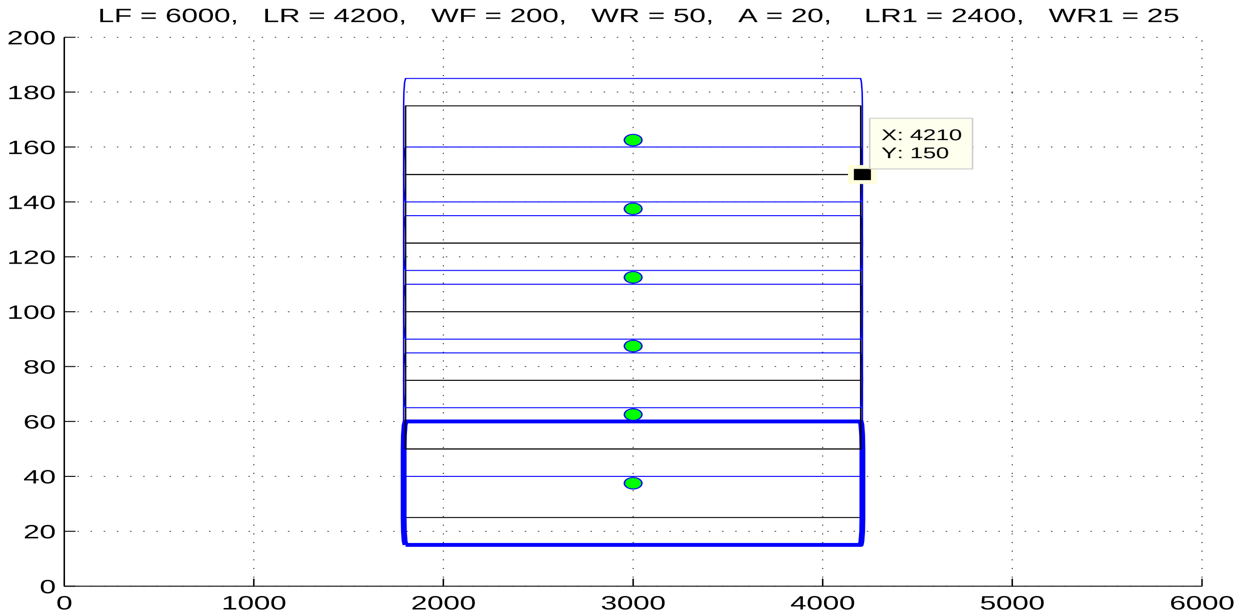

Figure 14. IC=5, C=5. A partitioning of F for data from [1], Table 3.

VII. SUMMARY

The purpose of this report is to use Monte Carlo procedures to produce software that supplies statistical solutions to a military problem which is beyond the desktop computer capabilities to utilize deterministic analyses.

A future study could be to adapt the developed software to the present-day problem of "interdicting" a river with a minimum of sensors (replacing the bombs in the present report). More specifically, determine the probability of successfully scanning the river for enemy moving objects with a minimum of appropriately placed sensors.

VIII. REFERENCES

1. M.P. Jarnagin Jr., *A Runway Interdiction Program*, NWL Technical Report TR-2413, March 1972, Naval Weapons Laboratory, Dahlgren, Virginia 22448.

2. A.H. Morris Jr., *NSWC Library of Mathematics Subroutines,* NSWCDD/TR-92/425, January 1993, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, Virginia 22448-5000.

3. A.M. Mood and F.A. Graybill, *Introduction to the Theory of Statistics,* pp. 251-263, McGraw Hill Book Co., New York, 1965.

4. A. Hald, *Statistical Theory with Engineering Applicatons,* pp. 697-699, John Wiley and Sons, New York, 1952.

5. M. Abramwoitz and I.A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs and Tables,* National Bureau of Standards Applied Mathematics Series, Vol. 55, U.S. Government Printing Office, 1964, Reprinted by Dover Publications, Inc., New York, 1965.

6. A.R. DiDonato and A.H. Morris Jr., *Significant Digit Computation of the Incomplete Beta Function Ratios,* ACM TOMS, Vol. 18, No. 3, September 1992, pp. 360-373.

7. A.R. DiDonato and M.P. Jarnagin, *A Method for Computing the Incomplete Beta Function Ratio,* NWL Technical Report No. 1949, Revised Edition, published 1966, Naval Weapons Laboratory, Dahlgren, Virginia 22448

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